Lec 16:

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Accretion of Plasma:

As we begin to consider the behavior of gases and plasmas under the influence of strong fields, a question that arises is whether a fluid description is applicable. A hydrodynamic approach is valid if the time scale for interactions between the constituent particles is shorter than that over which the external field changes the dynamics in the bulk.

The principal equations of hydrodynamics are formal representations of the conservation of mass, momentum and energy. The first equation of hydrodynamics describes the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

The second equation, which is the analogue of Newton's and law for an individual particle, states that:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial}{\partial x_j}\sigma_{ij} + \frac{1}{\delta V} F_{b,i}$$

Here δV is the volume of an element of fluid, $\vec{F_b}$ is the "body" force from an external field (like gravity), and σ_{ij} is the energy-momentum tensor. Note that σ_{ij} represents the i-th component of momentum density flux in the j-th direction. The terms on the right-hand side of the equation are "sources" that change the momentum of an element of fluid.

Finally, the third equation of hydrodynamics represents the conservation of the total energy:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + U + \Phi_g\right)\right] + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{1}{2}v^2 + U + \Phi_g\right)\right] = \rho T\frac{ds}{dt} - v_i \frac{\partial}{\partial x_j}(\sigma_{0,ij}) - \vec{\nabla}\cdot(P\vec{v})$$

Here P is the isotropic pressure, which is related to the energy-momentum tensor according to:

$$\sigma_{ij} = P\delta_{ij} + \sigma_{0,ij}$$

$\sigma_{0,ij}$ denotes the off-diagonal part of σ_{ij}. Also, u and s denote the internal energy and entropy per unit mass of the fluid, respectively, and ϕ_g is the gravitational potential. Again, the terms on the right-hand side of the equation act as sources that change the energy of an element of fluid. The first term, $\rho T \frac{ds}{dt}$, represents the heat exchange.

In most situation $\frac{\partial \phi_g}{\partial t} \simeq 0$, and also $\sigma_{0,ij} \simeq 0$. The third equation then takes a simpler form:

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u\right)\right] + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{1}{2}v^2 + u + \frac{P}{\rho} + \phi_g\right)\right] = \rho T\frac{ds}{dt}$$

With the help of the hydrodynamic equations, we can now discuss accretion of plasma onto compact objects.

Bondi-Hoyle Accretion:

We start by discussing radial accretion onto an isolated compact object. In reality, compact objects tend to be surrounded by an accretion disk as matter rarely falls radially. Nevertheless, spherical accretion is the simplest way of matter accreting, and we can learn much by considering it.

Let us rewrite the second and third hydrodynamic equations in slightly modified forms, which turn out to be more convenient:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial P}{\partial x_i} + f_{b,i}$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2 + u\right)\right] + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{1}{2}v^2 + u + \frac{P}{\rho}\right)\right] = -\vec{\nabla}\cdot\vec{F}_{rad} - \vec{\nabla}\cdot\vec{q} + \vec{v}\cdot\vec{f}_b$$

Here $\vec{f}_b = -\rho\vec{\nabla}\Phi_g$, $\vec{F}_{rad}$ is the radiative flux, and $\vec{q}$ is the convective flux.

We can often ignore $\vec{q}$. As for $\vec{F}_{rad}$, it depends on whether the accreting gas is optically thin or thick. In the optically thin limit, we have:

$$-\vec{\nabla}\cdot\vec{F}_{rad} = -4\pi\int j_\nu \, d\nu$$

Here j_ν is the emissivity. In the optically thick limit, $\vec{F}_{rad}$ is described by the blackbody law. The radiative force per unit volume is:

$$\vec{f}_{rad} = -\vec{\nabla} P_{rad} = -\frac{4}{3} a T^3 \vec{\nabla} T$$

The radiative momentum flux is given by $\frac{\vec{F}_{rad}}{c}$. If all of this momentum is absorbed within one mean free path ℓ, we have:

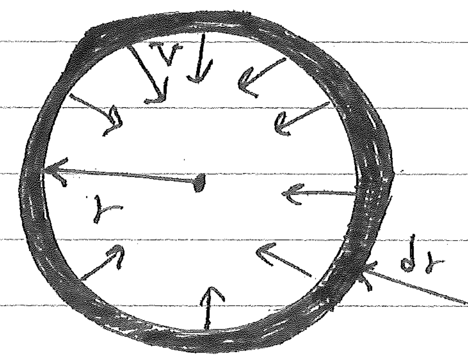
$$\frac{1}{\ell}\frac{\vec{F}_{rad}}{c} = -\frac{4}{3} a T^3 \vec{\nabla} T = \frac{-4ac}{3\kappa\rho} T^3 \vec{\nabla} T$$

Here we have used $\ell = \frac{1}{\kappa\rho}$, with κ being the opacity of the medium.

For a steady flow, all of the relevant quantities depend on the radial distance "r" only. The first equation of hydrodynamics then reads:

$$\frac{1}{r^2}\frac{d}{dr}(r^2 \rho v) = 0 \Rightarrow r^2 \rho v = \text{Const.}$$

The accretion rate is given by:



$$\dot{M} = -4\pi r^2 \rho v$$

Therefore the accretion rate is constant over time. Assuming that the mass of the accreting object changes very slowly, we have:

$$f_b = -\frac{GM\rho}{r^2}$$

The second hydrodynamics equation then becomes:

$$v\frac{dv}{dr} + \frac{1}{\rho}\frac{dP}{dr} + \frac{GM}{r^2} = 0$$

Assuming a polytropic relation $P = D\rho^{\Gamma}$ between the pressure and density of the accreting gas, we find:

$$\frac{dP}{dr} = \frac{dP}{d\rho}\frac{d\rho}{dr} = c_s^2 \frac{d\rho}{dr}$$

Where the sound speed is $c_s \equiv \left(\frac{dP}{d\rho}\right)^{\frac{1}{2}}$. Thus:

$$v\frac{dv}{dr} + \frac{c_s^2}{\rho}\frac{d\rho}{dr} = -\frac{GM}{r^2}$$

After using the equation for the conservation of mass, we find:

$$\frac{1}{2}\left(1-\frac{c_s^2}{v^2}\right)\frac{dv^2}{dr} = -\frac{GM}{r^2}\left(1-\frac{2c_s^2 r}{GM}\right)$$

This is called "Parker wind equation". Useful information can be obtained by looking at the asymptotic behavior of this equation. For accretion (i.e., inflow of gas), one expects that $\frac{dv^2}{dr} < 0$. As $r \to \infty$, the right-hand side of the equation becomes positive. As a result one must have $v < c_s$ as $r \to \infty$. This implies that the accretion is subsonic at large distances. On the other hand, the right-hand side is negative for $r \to 0$. Therefore $v > c_s$ in this limit, which implies supersonic accretion at small distances.

Transition from subsonic to supersonic $v = c_s$ happens at the point r_s, where:

$$1 - \frac{2c_s^2 r_s}{GM} = 0 \Rightarrow r_s = \frac{GM}{2c_s^2(r_s)}$$

With this insight, we now return to the equation obtained from the conservation of momentum:

$$v\frac{dv}{dr} + \frac{c_s^2}{\rho}\frac{d\rho}{dr} + \frac{GM}{r^2} = 0$$

From the polytropic relation we have $c_s^2 = D\Gamma\rho^{\Gamma-1}$, and $\frac{c_s^2}{\rho}\frac{d\rho}{dr} = \frac{D\Gamma}{\Gamma-1}\frac{d\rho^{\Gamma-1}}{dr}$. Integrating the above equation then yields:

$$\frac{1}{2}v^2 + \frac{D\Gamma}{\Gamma-1}\rho^{\Gamma-1} - \frac{GM}{r} = \text{Const.}$$

This is equivalent to:

$$\frac{1}{2}v^2 + \frac{c_s^2}{\Gamma-1} - \frac{GM}{r} = \text{Const.}$$

The constant value can be found by taking the $r \to \infty$ limit. In this limit $v \to 0$, $c_s \to c_s(\infty)$. The constant is therefore $\frac{c_s^2(\infty)}{\Gamma-1}$. As a result, we have:

$$\frac{1}{2}v^2 + \frac{c_s^2}{\Gamma-1} - \frac{GM}{r} = \frac{c_s^2(\infty)}{\Gamma-1}$$

At the point $r_s = \frac{GM}{2c_s^2}$, we have $v = c_s$. Thus:

$$c_s^2(r_s)\left(\frac{1}{2} + \frac{1}{\Gamma-1} - 2\right) = \frac{c_s^2(\infty)}{\Gamma-1}$$

This leads to:

$$c_s(r_s) = c_s(\infty)\left(\frac{2}{5-3\Gamma}\right)^{\frac{1}{2}}$$

The accretion rate is then found to be:

$$\dot{M} = 4\pi r_s^2 \rho(r_s)\, c_s(r_s)$$

Note that the polytropic relation implies:

$$\rho(r_s) = \rho(\infty)\left[\frac{c_s(r_s)}{c_s(\infty)}\right]^{\frac{2}{\Gamma-1}}$$

Finally, we find:

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{C_s^3(\infty)} \left(\frac{2}{5-3\Gamma}\right)^{\frac{5-3\Gamma}{2(\Gamma-1)}}$$

This is a remarkable relation that expresses the accretion rate in terms of the physical quantities at infinity.

Example: Consider an isolated neutron star accreting from the interstellar medium, where $\rho(\infty) \simeq 10^{-24}\ g\,cm^{-3}$ and $c_s(\infty) \simeq 10\,km\,s^{-1}$, corresponding to $T \simeq 10^4\,K$. The accretion rate of the neutron star is:

$$\dot{M} \simeq 1.4 \times 10^{11} \left(\frac{M}{M_\odot}\right)^2 \left[\frac{\rho(\infty)}{10^{-24}\ g\,cm^{-3}}\right] \left[\frac{C_s(\infty)}{10\,km\,s^{-1}}\right]^{-3} g\,s^{-1}$$